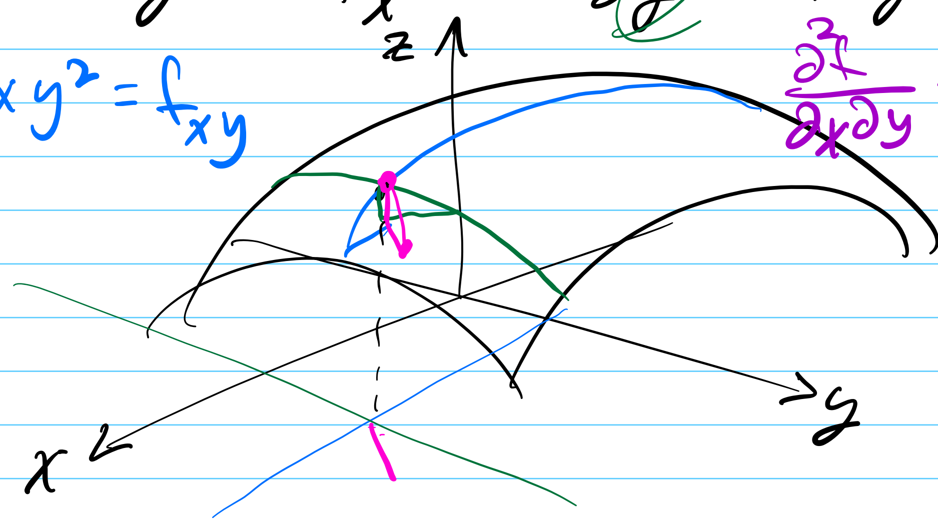
2.4 – Exact Equations

A partial derivative is an ordinary derivative holding all but one variable constant.

Notation: $f(x,y) = x^2y^3$

$\frac{\partial f}{\partial x} = 2xy^3 = f_x$ and $\frac{\partial f}{\partial y} = 3x^2y^2 = f_y$

$\frac{\partial^2 f}{\partial y \partial x} = 6xy^2 = f_{xy}$ $\qquad$ $\frac{\partial^2 f}{\partial x \partial y} = 6xy^2 = f_{yx}$



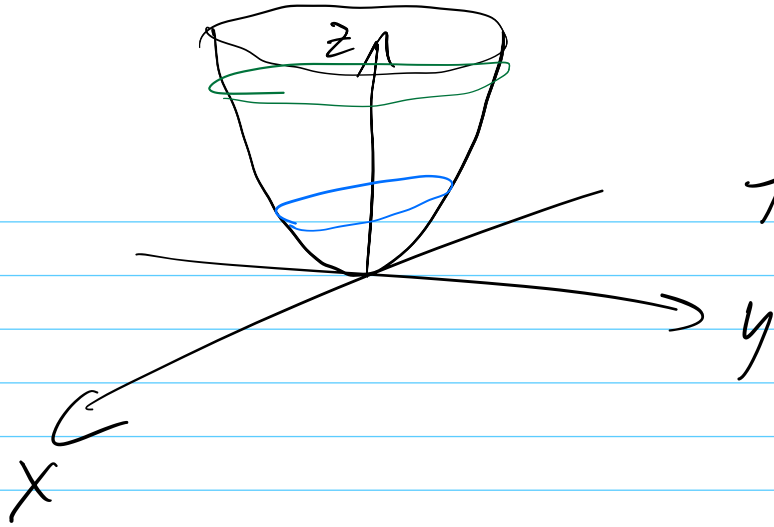
The full differential of f is

$$f_x\,dx + f_y\,dy$$

Consider $f(x,y) = C$ as a level curve to a surface.

In particular, if $f(x,y) = x^2 + y^2$,

$z = \boxed{f(x,y) = C}$ becomes $x^2 + y^2 = C$



Then $x^2+y^2=c$ is a level curve of this surface

Considering the level curve $f(x,y)=c$, the (full) exact differential is

$$f_x\,dx+f_y\,dy=0$$

For our work, we will consider $M\,dx+N\,dy=0$ the solution to which is $\boxed{f(x,y)=c}$.

$\int M\,dx$ $\qquad f(x,y) \qquad N=f_y$

$\frac{\partial f}{\partial x} \qquad \frac{\partial f}{\partial y}$

$M \qquad N$

$$\boxed{M_y \overset{?}{=} N_x} \quad \text{Test}$$

4. $(\sin y - y\sin x)\,dx + (\cos x + x\cos y - y)\,dy = 0$

$M = \sin y - y\sin x$ $\qquad$ $N = \cos x + x\cos y - y$

$M_y = \cos y - \sin x = N_x = -\sin x + \cos y$ ✓

$f(x,y) = \int M\,dx = \int (\sin y - y\sin x)\,dx$

$= x\sin y + y\cos x + h(y)$

$f_y = x\cos y + \cos x + h'(y) = \cos x + x\cos y - y = N$

$h'(y) = -y \Rightarrow h(y) = -\frac{1}{2}y^2 + k$

$f(x,y) = \boxed{x\sin y + y\cos x - \frac{1}{2}y^2 + k} = 0$

$$\boxed{x\sin y + y\cos x - \frac{1}{2}y^2 = C}$$

Form is $f(x,y) = c$

$M\,dx + N\,dy = 0$

8. $\left(1 + \ln x + \dfrac{y}{x}\right) dx = (1 - \ln x)\,dy$

$(1 + \ln x + \frac{y}{x})dx + (\ln x - 1)\,dy = 0$

Is it exact?

$M_y = \frac{1}{x}$ $\qquad$ $N_x = \frac{1}{x}$ $\qquad$ yes.

Ewww

$f(x,y) = \int M\,dx = \int \left(1 + \ln x + \frac{y}{x}\right) dx$

$f(x,y) = \int N\,dy = \int (\ln x - 1)\,dy$

$= y\ln x - y + g(x)$

$M = f_x : \quad 1 + \ln x + \frac{y}{x} = \frac{y}{x} + g'(x)$

$g'(x) = \int (1 + \ln x)\,dx$

$u = \ln x \quad dv = dx$
$du = \frac{1}{x} \quad v = x$

$g(x) = x + x\ln x - x + c$

$g(x) = x\ln x + c$

$f(x,y) = k \qquad \boxed{y\ln x + x\ln x - y = k}$

Form: $M \quad dx + \quad N \quad dy = 0$ ✓

22. $(e^x + y)\,dx + (2 + x + ye^y)\,dy = 0, \quad y(0) = 1$

Exact: $M_y = 1 \qquad N_x = 1$ ✓

$f(x,y) = \int (e^x + y)\,dx = e^x + xy + h(y)$

We know $N = f_y$

$2 + x + ye^y = x + h'(y)$

$h'(y) = 2 + ye^y$

$\int ye^y\,dy$

$u = y \quad dv = e^y dy$
$du = dy \quad v = e^y$

$$h(y) = \int h'(y)\,dy = 2y + ye^y - e^y + k$$

$f(x,y) = c$: $e^x + xy + 2y + ye^y - e^y = C$

$y(0) = 1 \Rightarrow$ when $x = 0$, $y = 1$

$$1 + 2 + e - e = c \Rightarrow c = 3$$

$$\boxed{e^x + xy + 2y + ye^y - e^y = 3}$$

Integrating factor:

In Problems 31–36 solve the given differential equation by finding, as in Example 4, an appropriate integrating factor.

34. $\cos x\, dx + \left(1 + \dfrac{2}{y}\right) \sin x\, dy = 0$

$M = \cos x$ $\qquad$ $N = \left(1 + \frac{2}{y}\right)\sin x$

$M_y = 0 \neq N_x = \left(1 + \frac{2}{y}\right)\cos x$

Not exact.

The integrating factor is either

$$\mu(x) = e^{\int \frac{M_y - N_x}{N}\,dx}$$

OR $\mu(y) = e^{\int \frac{N_x - M_y}{M}\,dy}$

$$\frac{M_y - N_x}{N} = -\frac{\cancel{\left(1+\frac{2}{y}\right)}\cos x}{\cancel{\left(1+\frac{2}{y}\right)}\sin x} = -\cot x$$

$$\mu(x) = e^{\int -\cot x \, dx}$$

$$\int -\cot x \, dx = -\int \frac{\cos x}{\sin x} dx = -\ln|\sin x|$$

$$\mu(x) = e^{\ln|\csc x|} = \csc x$$

$$\cos x \, dx + \left(1 + \frac{2}{y}\right) \sin x \, dy = 0$$

$$\cot x \, dx + \left(1 + \frac{2}{y}\right) dy = 0$$

$$f(x,y) = \int \cot x \, dx = \ln|\sin x| + h(y)$$

$$N_1 = f_y \Rightarrow \left(1 + \frac{2}{y}\right) = h'(y)$$

$$h(y) = y + \ln y^2 + k$$

$$\boxed{\ln|\sin x| + y + \ln y^2 = C}$$